

# Consistent theory for causal non-locality beyond Born's rule

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## Abstract

According to the theory of relativity and causality, a special type of correlation beyond quantum mechanics is possible in principle under the name of *non-local box*. The concept has been introduced from the principle of non-locality which satisfies relativistic causality. In this paper, we show that a correlation leading to the non-local box is possible to be derived consistently if we release the one of major axioms in quantum mechanics, *Born's rule*. This allows us to obtain a theory which in one end of the spectrum agrees with the classical probability and in the other end, agrees with the theory of non-local causality. At the same time, we argue that the correlation lies in a space with special mathematical constraints such that a physical realization of the correlation through a probability measure is not possible in one direction of its limit and is possible in the other limit.

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## I. INTRODUCTION

Quantum theory predicts a special type of correlation which allows an immediate action to take place on the state of a system at a distance [1]. Due to this special feature, the existence of extra ordinary correlations can be taken as a signature indicating whether a system behaves under the laws of quantum physics [2]. Even after extensive studies, the physical origin of the quantum correlation has not been unravelled. Specifically, the harmonious co-existence of this non-local quantum correlation with special relativity has been taken as the most challenging problem from the inception of the theories [3].

In quantum mechanics, there are axioms that lead us to a complete description of the theory [4]. Among them, *Born's rule* gives the probability that a measurement on a quantum system yields a particular result. The rule is named after Max Born, who interpreted the wave function of a state as a probability density which has become one of the key principles in quantum mechanics [5]. It provides a link between the mathematical formalism of quantum theory and the experimental realization of quantum measurement. The rule is responsible for practically all predictions of quantum physics. The statement of the rule is that if an observable  $\hat{X}$  with eigenstates  $\{|x_i\rangle\}$  is measured on a system described by a pure state  $|\psi\rangle$ , the probability that the measurement will yield the value  $x_i$  is given by

$$p(x_i) = |\langle x_i | \psi \rangle|^2. \quad (1)$$

where  $p(x_i)$  is the probability for the event of  $x_i$ .

Historically, numerous attempts have been made to derive Born's rule from first principles. In Gleason's theorem [6], Born's rule has been formulated from the basic mathematical assumptions for the probabilities of events as stated in Eq. (1). The probability of quantum mechanics is therefore dictated by the event structure generated from the propositions governing measurement [7]. However, the formulation does not necessarily provide justification about why nature chooses to behave as the rule describes. Deutsch tried to answer this question in an intuitive way [8]. He used the non-probabilistic axioms of quantum theory and classical decision theory to argue that the probabilities of quantum measurement outcomes can be derived as per Born's rule. The derivation sparked debates about the charge of circularity [9] and gave rise to new derivations from different angles, *e.g.* by Zurek [10]. However, the consensus remains that the precise place and of Born's rule among the axioms of quantum mechanics is not yet fully understood and continues to be questioned

[11]. Recently, an experimental test was performed on Born's rule through the exclusion of multi-order correlation [12, 13].

In this article, it is our intension to identify the implication of Born's rule on correlations in a bipartite system and show that a violation of quantum correlation can be obtained without Born's rule. box [14]. This is remarkable because lifting Born's rule offers a way of obtaining a generalized correlation that goes beyond quantum mechanics. We find that lifting Born's rule offers a way of obtaining a generalized correlation that goes beyond quantum mechanics, reaching a non-local box [14] as a limiting case. In our study, Born's rule is removed in such a way as to remain consistent with special relativity so that causal non-locality is still satisfied. Our observations allow us to conclude that without Born's rule, communication complexity can become trivial thus the theory becomes unphysical. We start our discussion by explaining the relationship between the theory of relativity and non-locality.

## II. CORRELATION FUNCTION FOR BELL'S INEQUALITY

In his historical lecture[3], Aharonov conjectured that *non-locality* and *relativistic causality* are the two main elements that specify quantum indeterminacy. Specifically, he argued that the non-local character of a quantum system can be regulated by special relativity as per the quantum correlation predicted by Bell [2] and Clauser-Horne-Shimony-Holt (CHSH) [15]. The CHSH function is of the form

$$\mathcal{B} = E(\vec{a}, \vec{b}) + E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}) - E(\vec{a}', \vec{b}') \quad (2)$$

where  $E(\vec{a}, \vec{b})$  is a correlation function between two parties. Considering a spin-1/2 bipartite system,  $E(\vec{a}, \vec{b})$  is defined as the measure of correlation of spins along the unit vectors  $\vec{a}$  and  $\vec{b}$ . Allocating the values +1 for spin up and -1 for spin down the correlation function can be written as a sum of joint probabilities

$$E(\vec{a}, \vec{b}) = p_{\uparrow\uparrow} + p_{\downarrow\downarrow} - p_{\uparrow\downarrow} - p_{\downarrow\uparrow} = p_{a=b} - p_{a \neq b} \quad (3)$$

where  $p_{a=b}$  and  $p_{a \neq b}$  refer to coincident and anti-coincident counts, respectively. Using the normalization condition  $p_{a=b} + p_{a \neq b} = 1$ , the correlation function becomes,

$$E(\vec{a}, \vec{b}) = 2p_{a=b} - 1 \quad (4)$$

and is bounded by  $-1 \leq E(\vec{a}, \vec{b}) \leq 1$  because  $0 \leq p_{a=b} \leq 1$ . Consequently, a simple algebraical consideration shows us that the function  $\mathcal{B}$  in Eq. (2) can take any arbitrary real values up to 4 without any constraints. However, an actual counting of local measurement outcomes does not allow the value of  $\mathcal{B}$  to exceed 2 [20]. In general,  $|\mathcal{B}| \leq 2$ . In fact, the local realistic model imposes a strong constraint on the joint probabilities given by classical spin systems. For a quantum mechanically correlated state of a spin-1/2 system, the maximal violation of the inequality goes up to  $2\sqrt{2}$ , called the Cirelson bound [16]. Aharonov conjectured that the bound is a consequence of special relativity.

However, it turns out that *non-locality* is a stronger notion of quantum statistics than *special relativity* [14]. It has been shown from the fact that the correlation which allows the violation of the Bell inequality over the Cirelson bound, say  $|\mathcal{B}| = 4$ , still satisfies the crucial constraint in the special relativity-*nothing can travel faster than the speed of light*. It implies that there can be a theory beyond quantum mechanics which satisfies special relativity.

A physical theory that accounts for a system bounded by the correlation  $2\sqrt{2} \leq |\mathcal{B}| \leq 4$  has never been properly formulated. This is partly because there is no known physical or non-physical theory governing correlations. In the following sections, we prove inductively that such a theory can be obtained once Born's rule is discarded.

### III. QUANTUM MECHANICAL CORRELATION FUNCTION

Based upon Gleason's theorem, a quantum mechanical formalism of correlations through the local measurements can be constructed for *observables*. The theory for the local measurements of spin-1/2 systems is given by the Pauli spin operators. Local measurement of a maximally entangled bipartite system in the singlet state  $|\psi^-\rangle$  gives the correlation function, as follows (see *e.g.* p 162 of [7])

$$\begin{aligned} E_q(\vec{a}, \vec{b}) &= \langle \psi^- | \vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma} | \psi^- \rangle \\ &= -\vec{a} \cdot \vec{b} = \cos(\theta_a - \theta_b). \end{aligned} \tag{5}$$

Comparing it with Eq.(4), we find  $p_{a=b} = \cos^2 \theta$  with the parameterization,  $\theta := |\theta_a - \theta_b|/2$ . Considering the underlying theory of the coincident probability, one can imagine a coincident probability amplitude  $\psi_{a=b}$  which generates the probability. In this case, Born's rule states that the square of the absolute value of the amplitude is the probability of the coincident

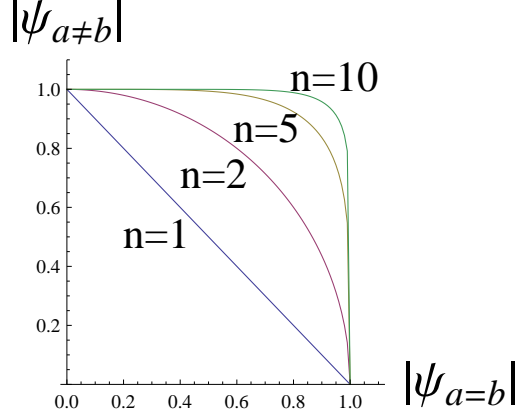


FIG. 1: The convexity of the probability amplitude under the normalization constraint  $|\psi_{a=b}|^n + |\psi_{a\neq b}|^n = 1$ . As  $n$  is increased, the function become more convex such that for a fixed  $|\psi_{a=b}|$  the value of  $|\psi_{a\neq b}|$  becomes larger as  $n$  grows.

counts :  $p_{a=b} = |\psi_{a=b}|^2$ . Due to the normalization condition  $|\psi_{a=b}|^2 + |\psi_{a\neq b}|^2 = 1$ , we arrive at the functional form of the probability amplitudes [21]

$$|\psi_{a=b}| = \cos \theta \quad \text{and} \quad |\psi_{a\neq b}| = \sin \theta. \quad (6)$$

Now, the probability amplitudes are parameterized by a single non-local parameter  $\theta$ . Here, we note that the correlation is a function of local measurement directions  $\theta_a$  and  $\theta_b$  only, since we consider the maximally entangled state to be tested. One can apply this to the CHSH function in Eq.(2) and the Cirelson bound of  $\mathcal{B} = 2\sqrt{2}$  is obtained when  $(\theta_a, \theta_b, \theta_{a'}, \theta_{b'}) = (0, \pi/4, \pi/2, 3\pi/4)$ .

#### IV. CORRELATION FUNCTION WITHOUT BORN'S RULE

Let us consider the consequence of discarding Born's rule. In general, Eq.(4) has to hold but the joint probability is no longer necessarily the absolute square of the amplitude. So, the correlation function can be written in general as

$$\begin{aligned} E_n(\theta_a, \theta_b) &= |\psi_{a=b}|^n - |\psi_{a\neq b}|^n \\ &= 2|\psi_{a=b}|^n - 1. \end{aligned} \quad (7)$$

where  $\theta_a$  and  $\theta_b$  are the local parameters specifying the local measurements. The second line uses the normalization condition that  $|\psi_{a=b}|^n + |\psi_{a\neq b}|^n = 1$ . It also means that the correlation

function can be subject to a single value parametrization whose physical meaning is directly linked to the angle between the local measurements at stations A and B. The probabilities of the coincidence and the anti-coincidence measurements for different  $n$  is plotted for this case in Fig.1. The convexity of the function increases for larger  $n$ , and becomes a step function in the limit of  $n \rightarrow \infty$ .

Motivated by the transformation from Cartesian to polar coordinates, one can define the angle  $\theta$  by  $\tan \theta = |\psi_{a=b}|/|\psi_{a \neq b}|$ ,  $0 < \theta < \pi/2$ . The correlation function then becomes  $E_n(\theta_a, \theta_b) = 1 - 2 \tan^n \theta / (1 + \tan^n \theta)$ . When  $n = 2$ , we have  $E_2(\theta) = E_q(\theta)$ . The non-local box can be obtained once  $E_\infty(\theta_a, \theta_b)$  and is constructed when  $\lim_{n \rightarrow \infty} \tan^n \theta = 0$  when  $0 < \theta < \pi/4$  and  $\lim_{n \rightarrow \infty} \tan^n \theta = \infty$  when  $\pi/4 < \theta < \pi/2$ . Generally, any theory producing correlation  $E_n(\theta_a, \theta_b)$  with integer  $n$ ,  $3 \leq n < \infty$ , implies the existence of a system which is asymptotically approaching the non-local box.

However, after a careful inspection, one realizes that the parametrization  $\tan \theta = |\psi_{a=b}|/|\psi_{a \neq b}|$  for the correlation function  $E_n(\theta)$  is not consistent with the local realistic model when  $n = 1$ . Therefore, the parameterization is not acceptable except for a quantum mechanical case with  $n = 2$ . The discrepancy occurs due to the convexity of the tangent function, which means that an increment of parameter  $\theta$  is not uniformly distributed over the change of the probability amplitudes  $|\psi_{a=b}|$  and  $|\psi_{a \neq b}|$ . In other words, the parameter does not produce the uniform distribution of the probability amplitudes  $|\psi_{a=b}|$  and  $|\psi_{a \neq b}|$  in the  $n$ -norm preserving space.

To satisfy consistency with a realistic model when  $n = 1$ , one should find the function  $F_n(\theta) := |\psi_{a=b}|^n$  that satisfies  $|\psi_{a=b}|^n + |\psi_{a \neq b}|^n = 1$  together with an extra condition,

$$\left( \partial |\psi_{a=b}| \right)^2 + \left( \partial |\psi_{a \neq b}| \right)^2 \propto (\partial \theta)^2 \quad (8)$$

which resembles the metric property in geometry. This condition, namely 2-norm uniformity condition, means that the displacement of the parameter  $\theta$  is uniformly distributed over the change of mutually exclusive probabilities. After some algebra [22], one realizes that the function  $F_n(\theta) = [G_n^{-1}(\theta)]^{1/n}$  can be found in a functional form of integration as  $G_n(x) = \frac{1}{n} \int dx [x^{-2(1-\frac{1}{n})} + (1-x)^{-2(1-\frac{1}{n})}]^{1/2}$ . Based upon the condition, the functional form of the correlation and the probability amplitude in Eq.(7) can be obtained as

$$E_n(\theta) = 2G_n^{-1}(\theta) - 1 \quad \text{and} \quad |\psi_{a=b}| = G_n^{-1}(\theta) \quad (9)$$

which reproduces generic theories from classic and quantum to non-local box. The correlation regulates the change of the Bell function in a way that the system never goes beyond the local realistic model when  $n = 1$  and reaches a quantum bound when  $n = 2$ . Analytic expressions of the correlation when  $n = 1, 2$  are

$$E_1(\theta) = 1 - \frac{4\theta}{\pi} \quad \text{and} \quad E_2(\theta) = \cos 2\theta \quad (10)$$

which coincide with the classical spin system and quantum mechanics. For the case of *classical* spin systems, a realistic model of the correlation is possible as described by  $E_1(\theta)$  [7] which follows from a realistic spin system existing in a unit sphere. In this setup, a spin measurement value is determined by cutting the equatorial plane of the sphere which is perpendicular to the measurement direction. The value of spin measurement is 1 for the spin pointing in one half of the sphere and is  $-1$  for the spin in the opposite side of the sphere. If a state of two classical spins is maximally correlated, it can be proven that the correlation is linearly proportional to the angle between the measurement directions of the two sides.

In general, the function  $G_n(x)$  with uniformity condition is not mathematically tractable. After numerical integration, the correlation with uniformity in Eq.(9) is plotted in Fig.2. In fact, the monotonic behavior of the correlation function coincides asymptotically with the function without the uniformity condition, although they are not equivalent. The difference between the correlation functions is less than 10% as shown in Fig.2. The two correlation functions coincide only when  $n = 2$  and  $n \rightarrow \infty$ . As it determines the functional form of the correlation uniquely, it is important to note that the uniformity condition is nonetheless trivial. The condition imposes a strong constraint on the correlation and *uniquely* determines the functional form of the correlation with respect to the measurement parameters.

Consequently, in the limit  $n \rightarrow \infty$ , the correlation leads to the non-local box. With a general property  $E_n(\pi/2 - \theta) = -E_n(\theta)$ , it can be seen that  $E_\infty(\theta) = 1$  for  $0 \leq \theta \leq \pi/4$  and  $E_\infty(\theta) = -1$  for  $\pi/4 \leq \theta \leq \pi/2$ . With four measurements separated by successive angle  $\pi/4$  as  $(\theta_a, \theta_b, \theta_{a'}, \theta_{b'}) = (0, \pi/4, \pi/2, 3\pi/4)$ , the CHSH function becomes

$$\begin{aligned} & E_\infty(\theta_a, \theta_b) + E_\infty(\theta_a, \theta_{b'}) + E_\infty(\theta_{a'}, \theta_b) - E_\infty(\theta_{a'}, \theta_{b'}) \\ &= 3E_\infty(\pi/8) - E_\infty(3\pi/8) = 4 \end{aligned} \quad (11)$$

which violates the CHSH inequality with the maximal value 4. The correlation function in the infinite power limit coincides with the one for the nonlocal box [14]. Therefore, we can

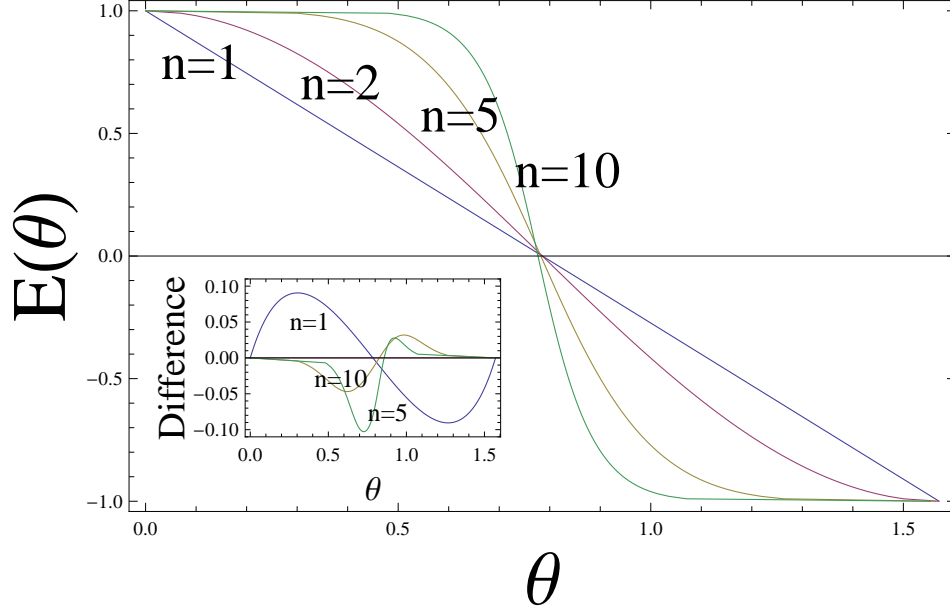


FIG. 2: Correlation function in Eq.(9) with uniformity condition for various  $n$  values. When  $n = 1$ , the correlation function is a straight line and when  $n = 2$ , the correlation function becomes a cosine function. As  $n$  increases, the correlation function approaches to the step function. Inset provides the differences between the correlation functions with and without the uniformity condition. The difference is within 10%. They coincide only when  $n = 2$  and  $n = \infty$ .

conclude that the correlation function  $E_n(\theta)$  is as general as to reproduce all the theories, classic, quantum and nonlocal box consistently.

## V. RELATIVISTIC CAUSALITY AND LOCALITY

An important question is whether the arbitrary  $n$ -norm preserving probability theory satisfies the assumption of relativistic causality. The fact is trivially true due to the normalization of the conditional probabilities *on the marginal distributions*. The conditions on the joint probabilities  $p(i, j|a, b)$  that the *no-signalling theorem* imposes are  $\sum_j p(i, j|a, b) = p(i|a)$  and  $\sum_i p(i, j|a, b) = p(j|b)$  where  $p(i|a)$  and  $p(j|b)$  are the conditional probabilities of local measurement  $a$  and  $b$  with outcomes  $i, j \in \{0, 1\}$  respectively. It means that the choice of measurement in one side does not affect the measurement probabilities in the other side. In quantum theory, the assumption is satisfied due to the completeness condition of the measurement projection [23].



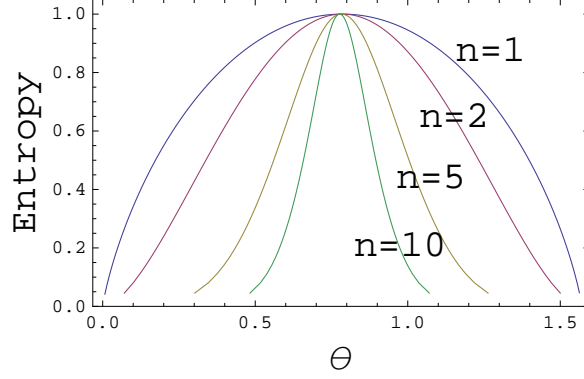


FIG. 3: Shannon entropy for the maximally correlated state of the  $n$ -norm probability theory. The correlation function (9) and relation (12) have been used for the calculation of the entropy.

Together with the condition of causality, the standard theory of probability and Bayesian law for a dichotomic system, one can derive the no-signalling condition whose joint probability of the local outcomes is given by

$$p(i, j|a, b) = \frac{p(i|a) + p(j|b) - 1/2}{2} + (-1)^{i+j} \frac{E_n(a, b)}{4} \quad (12)$$

where  $i, j \in \{0, 1\}$  and the correlation function  $E_n(\theta)$  in Eq.(9) has been used. The no-signalling condition always implies that the correlation function can be written in terms of the local probabilities with local parameters. (See the supplementary material for an extensive proof.)

Information that is contained in the new statistics can be represented by Shannon entropy  $S_n(\theta) = -\sum_{ij} p(i, j|a, b) \log p(i, j|a, b)$ , plotted in Fig.3 for a maximally correlated system, *i.e.*  $p(i|a) = p(j|b) = 1/2$  for  $\forall i, j$ . In that case, the convexity of the entropy changes with the measurement angle. When the spins are measured along the same direction, the system is completely certain,  $S_n(0) = 0$ , and when the measurements are orthogonal to each other, the local measurement outcomes of the two sides are completely random,  $S_n(\pi/4) = 1$ . The entropy becomes a concave function when  $n \geq 2$  from a convex function when  $n = 1$ . In the limit of  $n \rightarrow \infty$ , the entropy becomes a normalized delta function.

In the region of the *non-local causal space*, an important observation has been made through the fundamental principles of communication complexity [18]: it has been proven that any correlation function stronger than quantum mechanics would render all communication complexity problems *trivial* [18, 19]. Such an information theoretic implication has

been taken as strong evidence why a correlation cannot be stronger than quantum mechanics suggests. It means that  $n$  cannot take a value larger than 2 in a physical theory. On the other hand, it is worth stating that no physical constraint can be given in the region  $1 \leq n \leq 2$ , even for a non-integer  $n$ .

## VI. REMARK

*Remark* - Born's rule is one of the important axioms in quantum mechanics that connects most experimental observations to the theory. However, having discussed the non-local box in the framework of probability to show the strongest non-local correlation in a dichotomic bipartite system, we find that quantum mechanics under Born's rule never achieves such a strong correlation. In this Letter, we have derived a consistent correlation function as we discarded Born's rule under the constraint of the relativistic no-signalling condition. The correlation function  $E_n(\theta)$  in Eq.(9), is consistent with the local realistic model when  $n = 1$  and with quantum non-locality when  $n = 2$ . When  $n > 2$ , it shows stronger correlations than quantum non-locality, approaching to the non-local box when  $n \rightarrow \infty$ . We note that this is a direct consequence of releasing Born's rule which renders communication complexity trivial. Our study assures that Born's rule gives the physically meaningful correlations.

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- [20] Under the local realistic model, if one measures  $a = b$ ,  $a = b'$  and  $a' = b$ , then it is straightforward to know  $a' = b'$ . In that case,  $p_{a=b} = p_{a=b'} = p_{a'=b} = p_{a'=b'} = 1$  and we know  $\mathcal{B} = 2$ .
- [21] The precise derivation of the functional form is provided in the supplementary material.
- [22] From the completeness condition, the n-norm uniformity condition becomes  $dF_n/d\theta[1 + (1 - F_n^n)^{2(1/n-1)} F_n^{2(n-1)}]^{1/2} = \text{const.}$  and the substitution of  $F_n^n = x$  results that  $\int dx[x^{2(1/n-1)} + (1 - x)^{2(1/n-1)}]^2 = c\theta$  where  $c$  is an arbitrary constant. The constant takes the role of scaling

$\theta$ . The equation above implies that  $G[x] = c\theta$  where  $G[x] := \int dx [x^{2(1/n-1)} + (1-x)^{2(1/n-1)}]^2$  and it can be written that  $x = G^{-1}(\theta) = F_n^n(\theta)$ .

[23]  $\sum_j \text{Tr}[\hat{P}_i^A \otimes \hat{P}_j^B \rho_{AB}] = \text{Tr}[\hat{P}_i^A \rho_A]$  where  $\hat{P}_i^A$  and  $\hat{P}_j^B$  are orthogonal projectors at the sites A and B and  $\rho_A = \text{Tr}_B[\rho_{AB}]$ .